FINITENESS OF SIMPLE HOMOTOPY TYPE UP TO s-COBORDISM OF ASPHERICAL 4-MANIFOLDS

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Abstract. In this paper we show that for a large class $\mathcal{C}$ of 4-manifolds each member of $\mathcal{C}$ has only finitely many simple homotopy type up to s-cobordism. This result generalizes a similar result of Hillman for certain complex surfaces. We also present a correction in the proof of Hillman's result.

0. Introduction

The Borel conjecture says that closed aspherical manifolds are determined by their fundamental groups, i.e., an isomorphism between the fundamental groups of two closed aspherical manifolds is induced by a homeomorphism of the manifolds. In dimension greater than 4 this question is answered in positive for the class of manifolds with nonpositively curved Riemannian metric: this is the largest class of manifolds for which the answer is known so far. The advantage in higher dimension is the availability of the s-cobordism theorem and the surgery theory. In dimension 3 the answer is known for a vast class of manifolds: namely, Haken manifolds and hyperbolic manifolds. The answer will be complete in dimension 3 provided Thurston's Geometrization conjecture is true. The dimension 4 case is also not yet settled. For example the s-cobordism theorem and the exactness of the surgery sequence is known only for 4-manifolds with elementary amenable fundamental groups.

In this paper we show that for a class of aspherical 4-manifolds; up to s-cobordism there are only finitely many 4-manifolds simple homotopy equivalent to a given member of this class. Due to the unavailability of the 4-dimensional s-cobordism theorem we cannot quite say that there are only finitely many simple homotopy type up to homeomorphism of a manifold from this class.

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1. Finiteness of simple homotopy type

Let $\mathcal{C}$ be the class of compact orientable aspherical 4-manifolds so that for each $M \in \mathcal{C}$ there is a fiber bundle projection $M \to \mathbb{S}^1$ with irreducible fiber N and either M has nonempty boundary or it is closed and satisfies one of the following properties:

(1) $H^1(N, \mathbb{Z}) \neq 0$
(2) there is an embedded incompressible torus T in N so that $\pi_1(T)$ has square root closed image in $\pi_1(N)$
(3) N is a Seifert fibered space with base surface of genus ≥ 1
(4) N has more than 2 geometric pieces in its Jaco-Shalen and Johannson decomposition and the dual graph of this decomposition has a vertex which disconnects the graph and also the fundamental group of any edge emanating from this vertex is square root closed in the fundamental group of the target vertex
(5) N supports a hyperbolic metric.

Here recall that a subgroup H of a group G is called *square root closed* if for any $x \in G$, $x^2 \in H$ implies $x \in H$. And a 3-manifold is called *irreducible* if any embedded 2-sphere in it bounds an embedded 3-disc. An irreducible 3-manifold with nonempty boundary has nonvanishing first Betti number. Also any irreducible 3-manifold with nonvanishing first Betti number is Haken.

The dual graph of the JSJ-decomposition has vertices the pieces in the decomposition and edges are tori which are common boundary component of two pieces. By fundamental group of a vertex or an edge we mean the fundamental group of the associated spaces.

Before we state our main theorem we recall the definition of homotopy-cobordant structure sets: Let M be a compact manifold. Define $\mathcal{S}^s_{TOP}(M, \partial M) = \{(N, f) \mid f : N \to M$, where N a compact manifold, f a simple homotopy equivalence, $f|_{\partial N}$ is a homeomorphism onto $\partial M\ \}/ \simeq$, where $(N_1, f_1) \simeq (N_2, f_2)$ if there is a map $F : W \to M$ with domain W a s-cobordism with $\partial W = N_1 \cup N_2$ and $F|_{N_i} = f_i$. If the Whitehead group of $\pi_1(M)$ vanishes then this is the usual homotopy-topological structure set of M provided the s-cobordism theorem is true in $\dim M$. The dimension 4 s-cobordism theorem is known to be true only for 4-manifolds with elementary amenable fundamental group (see [FQ]).

In this paper we prove the following theorem:

Theorem 1.1. *Let $M \in \mathcal{C}$. Then the set $\mathcal{S}^s_{TOP}(M)$ is finite when M is closed and for $n \geq 1$ $\mathcal{S}^s_{TOP}(M \times \mathbb{D}^n, \partial(M \times \mathbb{D}^n))$ has only one element.*

Corollary 1.2. *Let $M \in \mathcal{C}$ and N be any other 4-manifold homotopy equivalent to M. Then there are integers r and s so that $M\#r(\mathbb{S}^2 \times \mathbb{S}^2)$ is diffeomorphic to $N\#s(\mathbb{S}^2 \times \mathbb{S}^2)$. Here $\#$ denotes connected sum. In such a case M and N are called stably diffeomorphic.*

Proof of Corollary 1.2. Follows from the main theorem in [D].

Remark 1.3. As it is not yet known if a s-cobordism between two 4-manifolds is trivial we cannot quite conclude that the manifolds in the class $\mathcal{C}$ has finitely many homotopy type up to homeomorphism.

Here we deduce an interesting corollary:

Corollary 1.4. *Let M be a nonsingular complex affine surface (i.e., a nonsingular complex algebraic surface in the complex space $\mathbb{C}^n$) which is a fiber bundle over the circle with irreducible (in 3-manifold sense) fiber. Then for $n \geq 1$ $M \times \mathbb{D}^n$ has only one homotopy type (with homotopy which are homeomorphism outside a compact set) up to homeomorphism.*

Proof. Using a suitable Morse function on M (for example consider the polynomial function $||x - x_0||^2$ for a fixed $x_0 \in M$) it is easily deduced (by Morse theory) that M is diffeomorphic to the interior of a compact aspherical 4-manifold. This follows because the restriction to M of any polynomial function has only finitely many critical value. (see corollary 2.8 in [M]). The Corollary follows. □

2. Proof of the theorem 1.1

At first we check that the fundamental group of any of the 4-manifolds in the class $\mathcal{C}$ has vanishing Whitehead group.

If ∂N is nonempty and has $\mathbb{S}^2$ as a boundary component then by irreducibility N is homeomorphic to $\mathbb{D}^3$ and hence M is homeomorphic to $\mathbb{D}^3 \times \mathbb{S}^1$. In this particular case the theorem is known. So we assume that if $\partial N \neq \emptyset$ then genus of any component of ∂N is ≥ 1.

Note that for any $M \in \mathcal{C}$ the fiber N of the fiber bundle $M \to \mathbb{S}^1$ is a Haken 3-manifold in the cases $(1) - (4)$. This implies $\pi_1(N) \in \mathcal{C}l$ from the notation of [W] and hence it has vanishing Whitehead group. Also $\pi_1(N)$ is regular coherent.

Now we can use the Mayer-Vietoris exact sequence (for K-theory) from [W] (Sections 17.1.3 and 17.2.3) to deduce that $\pi_1(M)$ has vanishing Whitehead group.

A general version of this fact is proved in (lemma V.3, [H1]).

If N is hyperbolic then by the Mostow rigidity theorem the monodromy diffeomorphism of the fiber bundle $M \to \mathbb{S}^1$ is homotopic to an isometry of finite order and hence $\pi_1(M)$ is isomorphic to the fundamental group of a 4-manifold M' which has a $\mathbb{H}^3 \times \mathbb{R}$ structure. Since M' is nonpositively curved $Wh(\pi_1(M)) = Wh(\pi_1(M')) = 0$ by [FJ]. This conclusion also can be made by noting that in fact the monodromy diffeomorphism is (topologically) isotopic (see [G] and [GMT]) to a (finite order) isometry and hence the fiber bundle M itself has $\mathbb{H}^3 \times \mathbb{R}$ structure.

In [Ro1] and [Ro2] we proved the following theorem:

Theorem 2.1. *(Theorem 1.2 in* [Ro1] *and Theorem 1.1 and 1.3 in* [Ro2]*) Let N be a compact orientable irreducible 3-manifold so that one of the following properties*

is satisfied:

(1) $H^1(N, \mathbb{Z}) \neq 0$

(2) *there is an embedded incompressible torus* T *in* N *so that the image of* $\pi_1(T)$ *is square root closed in* $\pi_1(N)$

(3) N *has more than* 2 *geometric pieces in its Jaco-Shalen and Johannson decomposition and the dual graph of this decomposition has a vertex which disconnects the graph and also the fundamental group of any edge emanating from this vertex is square root closed in the fundamental group of the target vertex.*

Then for $n \geq 2$ $N \times \mathbb{D}^n$ *has only one homotopy type up to homeomorphism.*

Here note that the case when N has nonempty boundary is included in case (1). In [Ro2] a large class of examples of 3-manifolds is given satisfying the property (2) and (3) in the above theorem. In fact it was shown there that if we consider the Jaco-Shalen and Johannson (JSJ) decomposition of the Haken manifold with T as one of the decomposing torus then the square root closed condition depends only on the pieces which abut the torus T. Also a large class of examples of 3-manifolds are given which has a square root closed incompressible torus boundary component.

We recall the Wall-Novikov surgery exact sequence here:

Let $\mathcal{S}(X, \partial X)$ denote the topological structure set of X of the group of homotopy type of X up to homeomorphism. For precise definition see any reference on surgery theory or in [Ro1]. (Here note that the differentiable structure set is not a group.) In terms of this group Theorem 2.1 say that $\mathcal{S}(N \times \mathbb{D}^n, \partial(\mathcal{S}(N \times \mathbb{D}^n))$ is trivial for $n \geq 2$. We always assume that $Wh(\pi_1(X)) = 0$. Then these groups fit into a long exact sequence of groups:

$$\cdots \longrightarrow \mathcal{S}_{n-1}(X) \longrightarrow H_n(X, \mathbb{L}_0) \longrightarrow L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X) \longrightarrow \cdots$$

Here $\mathcal{S}_n(X)$ are the total surgery obstruction group of Ranicki ([R1]) and they are in bijection with $\mathcal{S}(X \times \mathbb{D}^n, \partial(X \times \mathbb{D}^n))$ with a different indexing.

Note that the fundamental group of any $M \in \mathcal{C}$ is of the form $\pi_1(M) = \pi_1(N) \rtimes \mathbb{Z}$. From Theorem 2.1, the main theorem in [S] for the case (3) and by Farrell-Jones Topological Rigidity theorem for nonpositively curved Riemannian manifold (in the hyperbolic case) ([FJ]) it follows that the (assembly) map $H_n(N, \mathbb{L}_0) \longrightarrow L_n(\pi_1(N))$ is an isomorphism for large n.

Now the proof of the theorem follows from the following facts:

Fact 1: the Whitehead group of $\pi_1(M)$ vanishes.

Fact 2: the Ranicki Mayer-Vietoris exact sequence of surgery groups for groups which are semidirect product of a group with the infinite cyclic group (see [R2]).

Fact 3: the Mayer-Vietoris exact sequence of the generalized homology theory for $K(\pi, 1)$ spaces with coefficient in the surgery spectrum $\mathbb{L}_0$.

Fact 4: naturality of the assembly map and an application of five-lemma together with Theorem 2.1 and Siebenmann's periodicity theorem ([KS]).

Fact 5: the corollary to the theorem V.12 in [H1] which says that if the assembly map $H_5(M, \mathbb{L}_0) \to L_5(\pi_1(M))$ is an epimorphism then the set $\mathcal{S}^s_{TOP}(M)$ is finite. $\square$

Remark 2.2. Here we remark that the Theorem 1.1 will be true for all compact 4-manifolds which fiber over the circle if Thurston's conjecture is true: i.e., if any aspherical closed 3-manifold is either Haken, hyperbolic or Seifert fibered space, and if Theorem 2.1 is true for any Haken 3-manifold.

Remark 2.3. Hillman informed the author that he thinks it follows from [H3] that if M' is homotopically equivalent to $M \in \mathcal{C}$ then in fact M' and M are s-cobordant.

Remark 2.4. In [H1] Hillman proved Theorem 1.1 for the case when M also supports a complex structure. Also note that in the theorem the case of certain complex surfaces is included in case (3) because in ([H2]) it was proved if a complex surface fibers over the circle then the fiber is a Seifert fibered space. Hillman proved the Theorem 1.1 when the fiber is an arbitrary Seifert fibered space N assuming that N supports a nonpositively curved Riemannian metric (page 81 in [H1]). However, the unit tangent bundle of a closed oriented surface of genus ≥ 2 is a Seifert fibered space which has $\widetilde{\mathbb{SL}(2, \mathbb{R})}$ structure but does not support any nonpositively curved Riemannian metric (see [Ro1]).

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